

# On One-dimensional Multi-Particle Diffusion Limited Aggregation

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## Abstract

We prove that the one dimensional Multi-Particle Diffusion Limited Aggregation model has linear growth whenever the particle density exceeds 1 answering a question of Kesten and Sidoravicius. As a corollary we prove linear growth in all dimensions  $d$  when the particle density is at least 1.

## 1 Introduction

In the Diffusion Limited Aggregation (DLA) model introduced by Witten and Sanders [5] particles arrive from infinity and adhere to a growing aggregate. It produces beautiful fractal-like pictures of dendritic growth but mathematically it remains poorly understood. We consider a variant, multiparticle DLA, where the aggregate sits in an infinite Poisson cloud of particles which adhere when they hit the aggregate, a model which has been studied in both physics [4] and mathematics [2, 3]. Again one is interested in the growth of the aggregate and its structure.

In the model, initially there is a collection of particles whose locations are given by a mean  $K$  Poisson initial density on  $\mathbb{Z}^d$ . The particles each move independently according to rate 1 continuous time random walks on  $\mathbb{Z}^d$ . We follow the random evolution of an aggregate  $\mathcal{D}_t \subset \mathbb{Z}^d$  where at time 0 an aggregate is placed at the origin  $\mathcal{D}_0 = \{0\}$  to which other particles adhere according to the following rule. When a particle at  $v \notin \mathcal{D}_{t-}$  attempts to move onto the aggregate  $\mathcal{D}_t$  at time  $t$ , it stays in place and instead is added to the aggregate so  $\mathcal{D}_t = \mathcal{D}_{t-} \cup \{v\}$  and the particle no longer moves. Any other particles at  $v$  at the time are also frozen in place.

We will mainly focus on the one dimension setting and in Section 5 will discuss how to boost the results to higher dimensions. In this case the aggregate is simply a line segment and the processes on the positive and negative axes are independent so we simply restrict our attention to the rightmost position of the aggregate at time  $t$  which we denote  $X_t$ . In this case at time  $t$  when a particle at  $X_{t-} + 1$  attempts to take a step to the left it is incorporated into the aggregate along with any other particles.

It was proved by Kesten and Sidoravicius [2] that  $X_t$  grows like  $\sqrt{t}$  when  $K < 1$ . Indeed there simply are not enough particles around for it to grow faster. They conjectured, however, that when  $K > 1$  then it should grow linearly. Our main result confirms this conjecture.

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**Theorem 1** *For all  $K > 1$  the limit  $\lim_t \frac{1}{t} X_t$  exists almost surely and is a positive constant.*

We also give a simple extension of these results to higher dimensions and prove the following corollary.

**Corollary 2** *In all dimensions  $d \geq 2$  when  $K > 1$  the diameter of the aggregate grows linearly in  $t$ , that is for some positive constant  $\delta > 0$*

$$\lim_t \frac{1}{t} \text{Diam}(\mathcal{D}_t) > \delta \text{ a.s.}$$

Previously Sidoravicius and Stauffer [3] studied the case of  $d \geq 2$  in a slightly different variant where particles instead perform a simple exclusion process. They showed that for densities close to 1, that there is a positive probability that the aggregate grows with linear speed. Also in Section 5 we describe how for  $d \geq 2$  the upper bound on the threshold can be reduced further below 1, for example to  $\frac{5}{6}$  when  $d = 2$ . However, strikingly Eldan [1] conjectured that the critical value is always 0, that is the aggregate grows with linear speed for all  $K > 0$ . We are inclined to agree with this conjecture but our methods do not suggest a way of reaching the threshold. A better understanding of the growth of the standard DLA seems to be an important starting point.

## 2 Basic results

We will analyse the function valued process  $Y_t$  given by,

$$Y_t(s) := \begin{cases} X_t - X_{t-s} & 0 \leq s \leq t \\ \infty & s > t. \end{cases} \quad (1)$$

Let  $\mathcal{F}_t$  denote the filtration generated by  $X_t$ . We let  $S(t)$  denote the infinitesimal rate at which  $X_t$  increases given  $\mathcal{F}_t$ . Given  $\mathcal{F}_t$  the number of particles at  $X_t + 1$  is conditionally Poisson with intensity given by the probability that a random walker at  $X_t + 1$  at time  $t$  was never located in the aggregate. Each of the particles jumps to the left at rate  $\frac{1}{2}$  so with  $W_t$  denoting an independent continuous time random walk,

$$S(t) = \frac{1}{2} K \mathbb{P}[\max_{0 \leq s \leq t} W_s - Y_t(s) \leq 0 \mid Y_t].$$

Note that  $S(t)$  is an increasing as a function of  $Y_t$ . Indeed we could realise  $X_t$  as follows, let  $\Pi$  be a Poisson process on  $[0, \infty)^2$  and then

$$X_t = \Pi(\{(x, y) : 0 \leq x \leq t, 0 \leq y \leq S(x)\}).$$

Since both  $X_t$  and  $Y_t$  are increasing functions of  $\Pi$  we can make use of the FKG property. Also note that  $Y_t$  is stochastically decreasing.

Most of our analysis will involve estimating  $S(t)$  and using that to show that  $Y_t$  does not become too small for too long. Let  $M_t = \max_{0 \leq s \leq t} W_s$  be the maximum process of  $W_t$ .

**Lemma 2.1** *For any  $i \geq 0$  we have that*

$$S(t) \geq \frac{K}{2} \mathbb{P}[M_{2^i} = 0] \prod_{i'=i}^{\infty} \mathbb{P}[M_{2^{i'+1}} \leq Y_t(2^{i'}) \mid Y_t]$$

**Proof.** We have

$$\begin{aligned}
S(t) &\geq \frac{K}{2} \mathbb{P}[\max_{0 \leq s \leq t} W_s - Y_t(s) \leq 0 \mid Y_t] \\
&\geq \frac{K}{2} \mathbb{P}[M_{2^i} = 0, \forall i' \geq i \ M_{2^{i'+1}} \leq Y_t(2^{i'}) \mid Y_t] \\
&\geq \frac{K}{2} \mathbb{P}[M_{2^i} = 0] \prod_{i' \geq i} \mathbb{P}[M_{2^{i'+1}} \leq Y_t(2^{i'}) \mid Y_t]
\end{aligned}$$

where the final inequality follows from the FKG inequality.  $\square$

By the reflection principle we have that for any integer  $j \geq 0$ ,

$$\mathbb{P}[M_t \geq j] = \mathbb{P}[W_t \geq j] + \mathbb{P}[W_t \geq j+1].$$

Thus asymptotically we have that

$$\mathbb{P}[M_t = 0] \approx \frac{1}{\sqrt{2\pi}} t^{-1/2} \quad (2)$$

Now let  $T_j$  be the first hitting time of  $j$ . Since  $\cosh(s) - 1 \leq s^2$  for  $0 \leq s \leq 1$  we have that for  $t \geq 1$ ,

$$\mathbb{E}[e^{\frac{1}{\sqrt{t}} W_{t \vee T_j}}] \leq \mathbb{E}[e^{\frac{1}{\sqrt{t}} W_t}] = e^{(\cosh(\frac{1}{\sqrt{t}}) - 1)t} \leq e^1,$$

and hence by Markov's inequality

$$\mathbb{P}[M_t \geq jt^{1/2}] \leq \mathbb{P}[e^{\frac{1}{\sqrt{t}} W_{t \vee T_j}} = e^j] \leq e^{1-j}. \quad (3)$$

Plugging the above equations into Lemma 2.1 we get the following immediate corollary.

**Corollary 2.2** *There exists  $i^*$  such that the following holds. Suppose that  $i \geq i^*$  that for all  $i' \geq i$  we have  $j_{i'} = Y_t(2^{i'})2^{-i'/2}$ . Then*

$$S(t) \geq \frac{K}{10} 2^{-i'/2} \prod_{i'=i}^{\infty} (1 - e^{1 - \max\{1, j_{i'}/\sqrt{2}\}})$$

Next we check that provided  $S(t)$  remains bounded below during an interval then we get a comparable lower bound on the speed of  $X_t$ .

**Lemma 2.3** *We have that for all  $\rho \in (0, 1)$  there exists  $\psi(\rho) > 0$  such that for all  $\Delta > 0$ ,*

$$\mathbb{P}[\min_{s \in [t, t+\Delta]} S(s) \geq \gamma, X_{t+\Delta} - X_t \leq \rho \Delta \gamma \mid Y_t] \leq \exp(-\psi(\rho) \Delta \gamma)$$

*In the case of  $\rho = \frac{1}{2}$  we have  $\psi(\rho) \geq \frac{1}{10}$ .*

**Proof.** Using the construction of the process in terms of  $\Pi$  we have that

$$\begin{aligned}
\mathbb{P}[\min_{s \in [t, t+\Delta]} S(s) \geq \gamma, X_{t+\Delta} - X_t \leq \rho \Delta \gamma \mid Y_t] &\leq \mathbb{P}[\Pi([t, t+\Delta] \times [0, \gamma]) \leq \rho \Delta \gamma] \\
&= \mathbb{P}[\text{Poisson}(\Delta \gamma) \leq \rho \Delta \gamma]
\end{aligned}$$

Now if  $N \sim \text{Poisson}(\Delta\gamma)$  then  $\mathbb{E}e^{-\theta N} = \exp((e^{-\theta} - 1)\Delta\gamma)$  and so by Markov's inequality

$$\mathbb{P}[N \leq \rho\Delta\gamma] = \mathbb{P}[e^{-\theta N} \geq e^{-\theta\rho\Delta\gamma}] \leq \frac{\exp((e^{-\theta} - 1)\Delta\gamma)}{\exp(-\theta\rho\Delta\gamma)} = \exp((\theta\rho + e^{-\theta} - 1)\Delta\gamma).$$

Setting  $f_\rho(\theta) = -(\theta\rho + e^{-\theta} - 1)$  and

$$\psi(\rho) = \sup_{\theta \geq 0} f_\rho(\theta)$$

it remains to check that  $\psi(\rho) > 0$ . This follows from the fact that  $f_\rho(0) = 0$  and  $f'_\rho(0) = 1 - \rho > 0$ . Since  $f_{\frac{1}{2}}(\frac{1}{2}) \geq \frac{1}{10}$  we have that  $\psi(\frac{1}{2}) \geq \frac{1}{10}$ .  $\square$

### 3 Proof of Positive Speed

To measure our control over  $Y_t$  and show that it is moving quickly enough we say that  $Y_t$  is permissive at time  $t$  and at scale  $i$  if  $Y_t(2^i) \geq 10i2^{i/2}$ . Our approach, will be to consider functions

$$y_\alpha(s) = \begin{cases} 0 & s \leq \alpha^{-3/2} \\ \min\{\alpha(s - \alpha^{-3/2}), s^{1/2} \log_2 s\} & s \geq \alpha^{-3/2}. \end{cases}$$

and show that if  $Y_t(s) \geq y_\alpha(s)$  for increasing values of  $\alpha$  with good probability. To measure the speed of the aggregate in an interval of time define events  $\mathcal{R}$  as

$$\mathcal{R}(t, s, \gamma) = \{X_{t+s} - X_t \geq \gamma s\}.$$

**Lemma 3.1** *For all  $\epsilon > 0$  there exists  $0 < \alpha_\star(\epsilon) \leq 1$  such that for all  $0 < \alpha < \alpha_\star$ ,*

$$\mathbb{P}[\max_{s \geq 0} W_s - y_\alpha((s - \alpha^{-4/3}) \wedge 0) \leq 0] \geq 2(1 - \epsilon)\alpha.$$

**Proof.** For small  $\alpha_\star(\epsilon)$  we have that for  $\alpha^{-3/2} \leq s \leq \alpha^{-2}$ ,

$$\alpha(s - \alpha^{-4/3} - \alpha^{-3/2}) \leq (s - \alpha^{-4/3})^{1/2} \log_2(s - \alpha^{-4/3})$$

Hence with  $\xi = \xi_\alpha = \alpha^{4/3} + \alpha^{-3/2}$  if we set

$$\mathcal{A} = \{\max_{s \geq 0} W_s - \alpha((s - \xi) \wedge 0) \leq 0\}$$

and

$$\mathcal{B} = \{\max_{s \geq \alpha^{-2}} W_s - (s - \alpha^{-4/3})^{1/2} \log_2(s - \alpha^{-4/3}) \leq 0\}$$

then

$$\mathbb{P}[\max_{s \geq 0} W_s - y_\alpha((s - \alpha^{4/3}) \wedge 0) \leq 0] \geq \mathbb{P}[\mathcal{A}, \mathcal{B}] \geq \mathbb{P}[\mathcal{A}]\mathbb{P}[\mathcal{B}].$$

where the second inequality follows by the FKG inequality since  $\mathcal{A}$  and  $\mathcal{B}$  are both decreasing events for  $W_s$ . For large  $s$ , we have  $s^{1/2} \log_2 s \leq 2(s/2)^{1/2} \log_2(s/2)$  and so

$$\begin{aligned} \mathbb{P}[\mathcal{B}] &\geq \mathbb{P}[\max_{s \geq \alpha^{-2}} W_s - \frac{1}{2}s^{1/2} \log_2(\frac{1}{2}s) \leq 0] \\ &\geq \mathbb{P}[\forall i \geq \lfloor \log_2(\alpha^{-2}) \rfloor M_{2^{i+1}} \leq \frac{1}{2}(i-1)2^{i/2}] \\ &\geq \prod_{i \geq \lfloor \log_2(\alpha^{-2}) \rfloor} \mathbb{P}[M_{2^{i+1}} \leq \frac{1}{2}(i-1)2^{i/2}] \\ &\geq \prod_{i \geq \lfloor \log_2(\alpha^{-2}) \rfloor} e^{1-(i-1)2^{-3/2}} \end{aligned}$$

where the third inequality follows from the FKG inequality and the final inequality is by equation (3). Thus as  $\alpha \rightarrow 0$  we have that  $\mathbb{P}[\mathcal{B}] \rightarrow 1$  so it is sufficient to show that for small enough  $\alpha$  that  $\mathbb{P}[\mathcal{A}] \geq 2\alpha(1 - \epsilon/2)$ . By the reflection principle for  $a \leq 1$ ,

$$\mathbb{P}[M_t \geq 1, W_t = a] = \mathbb{P}[M_t \geq 1, W_t = 2 - a] = \mathbb{P}[W_t = 2 - a] = \mathbb{P}[W_t = a - 2],$$

and so

$$\begin{aligned} \mathbb{P}[M_t \geq 1] &= \sum_{a>1} \mathbb{P}[M_t \geq 1, W_t = a] + \sum_{a \leq 1} \mathbb{P}[M_t \geq 1, W_t = a] \\ &= \sum_{a>1} \mathbb{P}[M_t \geq 1, W_t = a] + \sum_{a \leq 1} \mathbb{P}[W_t = a - 2] \\ &= 1 - \mathbb{P}[W_t = 0] - \mathbb{P}[W_t = 1]. \end{aligned}$$

Hence by the Local Central Limit Theorem,

$$\lim_t \sqrt{t} \mathbb{P}[M_t = 0] = \lim_t \sqrt{t} (\mathbb{P}[W_t = 0] + \mathbb{P}[W_t = 1]) = \frac{2}{\sqrt{2\pi}}.$$

Also we have for  $a \leq 0$ ,

$$\mathbb{P}[W_t = a, M_t = 0] = \mathbb{P}[W_t = a] - \mathbb{P}[W_t = a, M_t \geq 1] = \mathbb{P}[W_t = a] - \mathbb{P}[W_t = a - 2]$$

and so the law of  $W_t$  conditioned on  $M_t = 0$  satisfies,

$$\begin{aligned} \lim_t \mathbb{P}\left[\frac{1}{\sqrt{t}} W_t \leq x \mid M_t = 0\right] &= \lim_t \frac{\sum_{a=-\infty}^{x\sqrt{t}} \mathbb{P}[W_t = a] - \mathbb{P}[W_t = a - 2]}{\mathbb{P}[M_t = 0]} \\ &= \lim_t \frac{\mathbb{P}[W_t = x\sqrt{t}] + \mathbb{P}[W_t = x\sqrt{t} - 1]}{\mathbb{P}[M_t = 0]} \\ &= \lim_t \frac{2 \frac{1}{\sqrt{2\pi}} e^{-x^2/2}}{\frac{2}{\sqrt{2\pi}}} = e^{-x^2/2} \end{aligned}$$

where  $x \leq 0$  and hence is the negative of the Rayleigh distribution. Now let  $Z_t = W_t - \alpha t$  and  $U_t = e^{\theta Z_t}$ . Then

$$\mathbb{E}U_t = \exp((\cosh(\theta) - 1 - \alpha\theta)t).$$

As  $f_\alpha(\theta) = \cosh(\theta) - 1 - \alpha\theta$  is strictly convex, it has two roots, one of which is at  $\theta = 0$ . Let  $\theta_\alpha$  be the non-zero root of  $f_\alpha$ . Since

$$f_\alpha(\theta) = -\alpha\theta + \frac{1}{2}\theta^2 + O(\theta^4)$$

for small  $\alpha$  we have that  $\theta_\alpha = 2\alpha + O(\alpha^2)$ . Then with  $\theta = \theta_\alpha$  we have that  $U_t = e^{\theta_\alpha Z_t}$  is a martingale. Let  $T = \min_t Z_t > 0$  and so by the Optional Stopping Theorem,

$$\mathbb{E}[U_T \mid Z_T = z] = \mathbb{E}[U_0 \mid Z_T = z] = e^{\theta_\alpha z}.$$

Also since  $U_T \in [0, 1]$  if  $T < \infty$  so

$$\mathbb{E}[U_T \mid Z_T = z] \geq \mathbb{P}[T < \infty \mid Z_T = z]$$

so

$$\mathbb{P}[T < \infty \mid Z_T = z] \leq e^{-\theta_\alpha}.$$

Thus we have that as  $\alpha \rightarrow 0$ ,

$$\begin{aligned} \mathbb{P}[\mathcal{A}] &= \mathbb{P}[\max_{s \geq 0} W_s - \alpha((s - \xi) \wedge 0) \leq 0] \\ &= \sum_{x=-\infty}^0 \mathbb{P}[M_\xi = 0, W_\xi = x] \mathbb{P}[T = \infty \mid Z_T = z] \\ &\geq \mathbb{P}[M_\xi = 0] \sum_{x=-\infty}^0 \mathbb{P}[W_\xi = x \mid M_\xi = 0] (1 - e^{\theta_\alpha x}) \\ &\geq \frac{2 + o(1)}{\sqrt{2\pi t}} \sum_{x=-\infty}^0 \mathbb{P}[W_\xi = x \mid M_\xi = 0] (-2\alpha x) \\ &\rightarrow \frac{4\alpha}{\sqrt{2\pi t}} \sqrt{\frac{\pi}{2}} = 2\alpha, \end{aligned}$$

since the mean of the Rayleigh distribution is  $\sqrt{\frac{\pi}{2}}$ . This completes the lemma.  $\square$

**Lemma 3.2** *For all  $K > 1$  there exists  $i_\star(K)$  such that if  $i \geq i_\star$  and  $Y_T$  is permissive at all levels  $i$  and above then with*

$$\alpha = \frac{1}{80} 2^{-i/2}$$

*we have that*

$$\mathbb{P}[\inf_s Y_{T+2^i}(s) - y_\alpha(s) \geq 0 \mid Y_T] \geq 1 - \exp(-2^{i/10}).$$

**Proof.** Since  $Y_T(2^{i'}) \geq 10i'2^{i'/2}$  for all  $i' \geq i$  if we set

$$\tilde{y}(s) = \begin{cases} 0 & s < 2^{i+1}, \\ 10j2^{j/2} & s \in [2^{j+1}, 2^{j+2}), j \geq i. \end{cases}$$

then since  $Y_{T+u}(s) \geq Y_T(s - u)$  then

$$\inf_{0 \leq u \leq 2^i} \inf_{s \geq 0} Y_{T+u}(s) - \tilde{y}(s) \geq 0. \quad (4)$$

By Corollary 2.2 for all  $t \in [0, 2^i]$

$$S(t) \geq \frac{1}{10} 2^{-(i+1)/2} \prod_{i'=i}^{\infty} (1 - e^{1 - \max\{1, 5(i'-1)\}}) \geq \frac{1}{20} 2^{-i/2},$$

where the second inequality holds provided that  $i_\star(K)$  is sufficiently large. Defining  $\mathcal{D}$  as the event that  $X_t$  moves at rate at least  $\frac{1}{40} 2^{-i/2}$  for each interval  $\ell 2^{2i/3}, (\ell + 1) 2^{2i/3}$ ,

$$\mathcal{D} = \bigcap_{\ell=0}^{2^{i/3}-1} \mathcal{R}(T + \ell 2^{2i/3}, 2^{2i/3}, \frac{1}{40} 2^{-i/2})$$

by Lemma 2.3 we have that

$$\mathbb{P}[\mathcal{D}] \geq 1 - 2^{i/3} \exp(-\frac{1}{10} \cdot 2^{2i/3} \cdot \frac{1}{40} 2^{-i/2}) \geq 1 - \exp(-2^{i/10})$$

where the last inequality holds provided that  $i_*(K)$  is sufficiently large. We claim that on the event  $\mathcal{D}$ , we have that  $Y_{T+2^i}(s) \geq y_\alpha(s)$  for all  $s$ . For  $s \geq 2^{i+1}$  this holds since by equation (4) we have that

$$Y_{T+2^i}(s) \geq \tilde{y}(s) \geq s^{1/2} \log_2 s \geq y_\alpha(s).$$

For  $0 \leq s \leq 2^i$ , on the event  $\mathcal{D}$ ,

$$Y_{T+2^i}(s) \geq \lfloor s 2^{-2i/3} \rfloor 2^{2i/3} \frac{1}{40} 2^{-i/2} \geq \max\{0, s - \alpha^{-3/2}\} \frac{1}{40} 2^{-i/2} \geq y_\alpha(s),$$

and for  $2^i \leq s \leq 2^{i+1}$

$$Y_{T+2^i}(s) \geq Y_{T+2^i}(2^i) \geq 2^i \cdot \frac{1}{40} 2^{-i/2} \geq y_\alpha(2^{i+1}).$$

Thus for all  $s \geq 0$ ,  $Y_{T+2^i}(s) \geq y_\alpha(s)$  which completes the proof.  $\square$

**Lemma 3.3** *For all  $K > 1$ , there exists  $\Delta(K)$  and  $\chi(K) > 0$  such that if  $0 \leq \alpha \leq \Delta$  and  $\inf_s Y_T(s) - y_\alpha(s) = 0$  then*

$$\mathbb{P}\left[\mathcal{R}(T, \alpha^{-4/3}, \frac{\alpha(K+1)}{2})^c \mid Y_T\right] \leq \exp\left(-\chi(K)\alpha^{-1/3}\right).$$

**Proof.** With  $\alpha_*(\epsilon)$  defined as in Lemma 3.1 set  $\Delta(K) = \alpha_*(\frac{K-1}{3K})$ . Then for  $0 \leq s \leq \alpha^{-4/3}$

$$\begin{aligned} S(T+t) &= \frac{K}{2} \mathbb{P}[\max_{0 \leq s \leq t} W_s - Y_{T+t}(s) \leq 0 \mid Y_{T+t}] \\ &\geq \frac{K}{2} \mathbb{P}[\max_{s \geq 0} W_s - y_\alpha((s - \alpha^{4/3}) \wedge 0) \leq 0] \\ &\geq \frac{K}{2} 2\left(1 - \frac{K-1}{3K}\right)\alpha = \frac{\alpha(2K+1)}{3} \end{aligned}$$

where the first inequality follows from the fact that

$$Y_{T+t}(s) \geq Y_T(s - \alpha^{4/3}) \wedge 0 \geq y_\alpha(s - \alpha^{4/3}) \wedge 0$$

and the second inequality follows from Lemma 3.1. Now take  $\rho = \frac{3K+3}{4K+2} < 1$  and with  $\psi$  defined in Lemma 2.3 set  $\chi(K) = \psi(\rho)$ . Then since

$$\inf_{0 \leq t \leq \alpha^{-4/3}} S(T+t) \geq \frac{\alpha(2K+1)}{3} = \rho \frac{\alpha(K+1)}{2}$$

by Lemma 2.3 we have that

$$\mathbb{P}\left[\mathcal{R}(T, \alpha^{-4/3}, \frac{\alpha(K+1)}{2})^c \mid Y_T\right] \leq \exp\left(-\chi(K)\alpha^{-1/3}\right).$$

$\square$

This result is useful because of the following claim.

**Claim 3.4** *For some  $0 \leq \alpha \leq \frac{1}{2}$  suppose that  $\inf_s Y_T(s) - y_\alpha(s) = 0$ . Then for an  $0 \leq t \leq \alpha^{-3/2}$  and  $\gamma \geq 1$  on the event  $\mathcal{R}(T, t, \alpha\gamma)$  we have that  $\inf_s Y_{T+t}(s) - y_\alpha(s) = 0$ .*

**Proof.** Since  $y_\alpha(s) =$  for  $0 \leq s \leq \alpha^{-3/2}$  it is sufficient to check  $s \geq \alpha^{-3/2}$ . Then

$$\begin{aligned} Y_{T+t}(s) &= Y_T(s-t) + X_{T+t} - X_t \\ &\geq Y_T(s-t) + \alpha\gamma t \\ &\geq y_\alpha(s-t) + \alpha\gamma t \\ &\geq y_\alpha(s) - \alpha t + \alpha\gamma t \geq y_\alpha(t), \end{aligned}$$

where the first inequality is by the event  $\mathcal{R}(T, t, \alpha\gamma)$ , the second is by assumption and the third is since  $\frac{d}{ds}y_\alpha(s)$  is uniformly bounded above by  $\alpha$ .  $\square$

**Lemma 3.5** *For all  $K > 1$ , there exists  $\Delta(K)$  and  $\chi(K) > 0$  such that if  $0 \leq \alpha \leq \Delta$  and  $\inf_s Y_T(s) - y_\alpha(s) = 0$  then*

$$\mathbb{P}\left[\inf_s Y_{T+\alpha^{-3}}(s) - y_{\frac{\alpha(K+1)}{2}}(s) = 0 \mid Y_T\right] \leq \alpha^{-5/3} \exp\left(-\chi(K)\alpha^{-1/3}\right).$$

**Proof.** Let  $\mathcal{D}_\ell$  denote the event,

$$\mathcal{D}_\ell = \mathcal{R}(T + \ell\alpha^{-4/3}, \alpha^{-4/3}, \frac{\alpha(K+1)}{2}).$$

By Claim 3.4 and induction if  $\bigcap_{\ell'=0}^{\ell-1} \mathcal{D}_{\ell'}$  holds then  $\inf_s Y_{T+\ell\alpha^{-4/3}}(s) - y_\alpha(s) = 0$ . Thus by Lemma 3.3 we have that

$$\mathbb{P}\left[\mathcal{D}_\ell \mid \bigcap_{\ell'=0}^{\ell-1} \mathcal{D}_{\ell'}, Y_T\right] \geq 1 - \exp\left(-\chi(K)\alpha^{-1/3}\right)$$

and so with  $\mathcal{D}^* = \bigcap_{\ell=0}^{\alpha^{-5/3}-1} \mathcal{D}_\ell$ ,

$$\mathbb{P}[\mathcal{D}^* \mid Y_T] \geq 1 - \alpha^{-5/3} \exp\left(-\chi(K)\alpha^{-1/3}\right).$$

Now suppose that the event  $\mathcal{D}^*$  holds and assume that  $\Delta(K)$  is small enough so that for all  $0 \leq \alpha \leq \Delta(K)$  the following hold:

- $\alpha^{-4/3} \leq (\frac{\alpha(K+1)}{2})^{-3/2}$ ,
- $\frac{K+1}{2}\alpha^{-2} \geq (2\alpha^{-3})^{1/2} \log_2(2\alpha^{-3})$ ,
- $\forall s \geq \alpha^{-3}, \min\{\alpha(s - \alpha^{-3/2}), s^{1/2} \log_2 s\} = s^{1/2} \log_2 s$ ,
- $\forall s \geq \alpha^{-3}, \min\{\frac{\alpha(K+1)}{2}(s - (\frac{\alpha(K+1)}{2})^{-3/2}), s^{1/2} \log_2 s\} = s^{1/2} \log_2 s$ ,
- $\inf_{s \geq 2\alpha^{-3}} -s^{1/2} \log_2 s + (s - \alpha^{-3})^{1/2} \log_2(s - \alpha^{-3}) + \frac{K+1}{2}\alpha^{-2} \geq 0$ .

It is straightforward to check that all of these hold for sufficiently small  $\alpha$ . For all  $(\frac{\alpha(K+1)}{2})^{-3/2} \leq s \leq \alpha^{-3}$  that

$$Y_{T+\alpha^{-3}}(s) \geq \lfloor s\alpha^{4/3} \rfloor \alpha^{-4/3} \cdot \frac{\alpha(K+1)}{2} \geq \left(s - \left(\frac{\alpha(K+1)}{2}\right)^{3/2}\right) \frac{\alpha(K+1)}{2} \geq y_{\frac{\alpha(K+1)}{2}}(s).$$



For  $\alpha^{-3} \leq s \leq 2\alpha^{-3}$ ,

$$\begin{aligned} Y_{T+\alpha^{-3}}(s) &\geq Y_T(s - \alpha^{-3}) + \frac{K+1}{2}\alpha^{-2} \\ &\geq y_\alpha(s - \alpha^{-3}) + \frac{K+1}{2}\alpha^{-2} \\ &\geq (2\alpha^{-3})^{1/2} \log_2(2\alpha^{-3}) = y_{\frac{\alpha(K+1)}{2}}(2\alpha^{-3}). \end{aligned}$$

Finally, for  $s \geq 2\alpha^{-3}$ ,

$$\begin{aligned} Y_{T+\alpha^{-3}}(s) &\geq y_\alpha(s - \alpha^{-3}) + \frac{K+1}{2}\alpha^{-2} \\ &= y_{\frac{\alpha(K+1)}{2}}(s) - s^{1/2} \log_2 s + (s - \alpha^{-3})^{1/2} \log_2(s - \alpha^{-3}) + \frac{K+1}{2}\alpha^{-2} \\ &\geq y_{\frac{\alpha(K+1)}{2}}(s). \end{aligned}$$

Combining the previous 3 equations implies that  $Y_{T+\alpha^{-3}}(s) \geq y_{\frac{\alpha(K+1)}{2}}(s)$  for all  $s$  and hence

$$\mathbb{P}\left[\inf_s Y_{T+\alpha^{-3}}(s) - y_{\frac{\alpha(K+1)}{2}}(s) = 0 \mid Y_T\right] \leq \mathbb{P}[\mathcal{D}^*] \leq \alpha^{-5/3} \exp\left(-\chi(K)\alpha^{-1/3}\right).$$

□

**Lemma 3.6** *For all  $K > 1$ , there exists  $i^*(K)$  such that the following holds. If  $i \geq i^*$  and  $Y_T$  is permissive for all  $i' > i$  then*

$$\mathbb{P}\left[\min_{s \in [4^i, 2e^{2^{i/10}}]} Y_{T+s}(2^i) \leq 10i2^{i/2} \mid \mathcal{F}_T\right] \leq 3e^{-2^{i/10}},$$

that is  $Y_{T+s}$  is permissive at scale  $i$  for all  $s \in [2^i, 2e^{2^{i/10}}]$ .

**Proof.** We choose  $i^*(K)$  large enough so that,

$$20i^* K 2^{-i^*/2} \leq \Delta(K)$$

where  $\Delta(K)$  was defined in 3.5. Set  $t_0 = 2^i$  and  $\alpha_0 = \frac{1}{80}2^{-i/2}$ . We define  $\alpha_\ell = \left(\frac{K+1}{2}\right)^\ell \alpha_0$  and  $t_\ell = t_{\ell-1} + \alpha_{\ell-1}^{-3}$ . Define the event  $\mathcal{W}_\ell$  as

$$\mathcal{W}_\ell = \left\{ \inf_s Y_{T+t_\ell}(s) - y_{\alpha_\ell}(s) = 0 \right\}.$$

By Lemma 3.2 we have that

$$\mathbb{P}[\mathcal{W}_0 \mid \mathcal{F}_T] \geq 1 - \exp(-2^{i/10}),$$

and by Lemma 3.5 we have that

$$\mathbb{P}\left[\mathcal{W}_\ell \mid \bigcap_{\ell'=0}^{\ell-1} \mathcal{W}_{\ell'} \mid \mathcal{F}_T\right] \geq 1 - \alpha_{\ell-1}^{-5/3} \exp\left(-\chi(K)\alpha_{\ell-1}^{-1/3}\right).$$

Now choose  $L$  to be the smallest integer such that  $\alpha_L \geq 20i2^{-i/2}$ . So  $L = \lceil \frac{\log(1600i)}{\log((K+1)/2)} \rceil$  which is bounded above by  $i$  provided that  $i^*(K)$  is sufficiently large. Thus

$$\begin{aligned} \mathbb{P}[\mathcal{W}_L \mid \mathcal{F}_T] &\geq 1 - \exp(-2^{i/10}) - \sum_{\ell=0}^{L-1} \alpha_{\ell-1}^{-5/3} \exp\left(-\chi(K)\alpha_{\ell-1}^{-1/3}\right) \\ &\geq 1 - \exp(-2^{i/10}) - i(20i2^{-i/2})^{-5/3} \exp\left(-\chi(K)(20i2^{-i/2})^{-1/3}\right) \\ &\geq 1 - 2\exp(-2^{i/10}) \end{aligned}$$

where the final inequality holds for  $i$  is sufficiently large. Now let  $\mathcal{D}_k$  denote the event,

$$\mathcal{D}_k = \mathcal{R}(T + t_L + k\alpha_L^{-4/3}, \alpha_L^{-4/3}, \alpha_L).$$

By Claim 3.4 on the event  $\mathcal{W}_L$  and  $\bigcap_{k'=0}^{k-1} \mathcal{D}_{k'}$  we have

$$\inf_s Y_{T+t_L+k\alpha_L^{-4/3}}(s) - y_{\alpha_L}(s) = 0.$$

Thus by Lemma 3.3 we have that

$$\mathbb{P}[\mathcal{D}_k \mid \mathcal{F}_T, \mathcal{W}_L, \bigcap_{k'=0}^{k-1} \mathcal{D}_{k'}] \geq 1 - \exp\left(-\chi(K)\alpha_L^{-1/3}\right).$$

Let  $\mathcal{D}^*$  be the event

$$\mathcal{D}^* = \left\{ \mathcal{W}_L, \bigcap_{k'=0}^{e^{2^{i/10}}-1} \mathcal{D}_{k'} \right\}.$$

Then for  $i$  sufficiently large since  $\alpha_L \leq 20iK2^{-i/2}$ ,

$$\mathbb{P}[\mathcal{D}^* \mid \mathcal{F}_T] \geq 1 - 2\exp(-2^{i/10}) - \exp\left(2^{i/10} - \chi(K)\alpha_L^{-1/3}\right) \geq 1 - 3\exp(-2^{i/10}).$$

On the event  $\mathcal{D}^*$  we have that for all  $t_L + 2^i \leq s \leq \alpha_L^{-4/3} e^{2^{i/10}}$  that

$$Y_{T+s} \geq \alpha_L(2^i - \alpha_L^{-4/3}) \geq 10i2^{i/2}.$$

By construction  $t_L = 2^i + \sum_{\ell=0}^{L-1} \alpha_{\ell}^{-3} \leq 4^i$  and hence

$$\mathbb{P}\left[\min_{s \in [4^i, 2e^{2^{i/10}}]} Y_{T+s}(2^i) \leq 10i2^{i/2} \mid \mathcal{F}_T\right] \leq \mathbb{P}[(\mathcal{D}^*)^c \mid \mathcal{F}_T] \leq 3e^{-2^{i/10}}.$$

□

**Corollary 3.7** *For all  $K > 1$ , there exists  $i^*(K)$  such if  $i \geq i^*$  then*

$$\mathbb{P}\left[\min_{s \in [0, e^{2^{i/10}}]} Y_s(2^i) \leq 10i2^{i/2}\right] \leq 3e^{-2^{i/10}},$$

**Proof.** We can apply Lemma 3.6 to time  $T = 0$  since it is permissive at all levels and hence have that

$$\mathbb{P}\left[\min_{s \in [4^i, 2e^{2^{i/10}}]} Y_s(2^i) \leq 10i2^{i/2}\right] \leq 3e^{-2^{i/10}},$$

Since  $Y_t$  is stochastically decreasing in  $t$  we have that

$$\mathbb{P}\left[\min_{0 \leq t \leq e^{2^{i/10}}} Y_t(2^i) \leq 10i2^{i/2} - i2^i\right] \leq \mathbb{P}\left[\min_{s \in [4^i, 4^i + e^{2^{i/10}}]} Y_s(2^i) \leq 10i2^{i/2}\right] \leq 3e^{-2^{i/10}},$$

which completes the corollary.  $\square$

**Lemma 3.8** *For all  $K > 1$ , there exists  $i^*(K)$  such that*

$$\inf_t \mathbb{P}\left[\forall i \geq i^*, Y_t(2^i) \geq 10i2^{i/2}\right] \geq \frac{1}{2}.$$

**Proof.** Take  $i^*(K)$  as in Lemma 3.6 and suppose that  $I \geq i^*$ . Let  $\mathcal{D}_I$  denote the event that  $Y_t$  is permissive for all levels  $i \geq I$  and all  $t \in [0, e^{2^{I/10}}]$ . By Corollary 3.7 we have that

$$\mathbb{P}[\mathcal{D}_I^c] \leq \sum_{i \geq I} 3e^{-2^{i/10}} \leq 4e^{-2^{I/10}}.$$

Next set  $t_0 = \frac{1}{2}e^{2^{I/10}}$  and let  $t_k = t_{k-1} + 4^{I-k}$ . Let  $\mathcal{H}_k$  denote the event that  $Y_t$  is permissive at level  $I - k$  for all  $t \in [t_k, t_k + e^{2^{(I-k)/10}}]$ . By Lemma 3.6 then for  $0 \leq k \leq I - i^*$ ,

$$\mathbb{P}[\mathcal{H}_k^c, \cap_{k'=1}^{k-1} \mathcal{H}_{k'}, \mathcal{D}_{i^*}] \leq 3e^{-2^{(I-k)/10}}.$$

Thus, provided  $i^*$  is large enough,

$$\mathbb{P}[\cap_{k'=1}^{I-i^*} \mathcal{H}_{k'}, \mathcal{D}_{i^*}] \geq 1 - 4e^{-2^{I/10}} - \sum_{k'=1}^{I-i^*} 3e^{-2^{(I-k)/10}} \geq \frac{1}{2}.$$

Let  $\tau = \tau_I = t_{I-i^*}$ . Then for all  $I \geq i^*$ ,

$$\mathbb{P}\left[\forall i \geq i^*, Y_{\tau_I}(2^i) \geq 10i2^{i/2}\right] \geq \delta.$$

since  $Y_t$  is stochastically decreasing in  $t$  and  $\tau_I \rightarrow \infty$  and  $I \rightarrow \infty$ ,

$$\inf_t \mathbb{P}\left[\forall i \geq i^*, Y_t(2^i) \geq 10i2^{i/2}\right] \geq \frac{1}{2}.$$

$\square$

**Theorem 3.9** *For  $K > 1$  there exists a random function  $Y^*(s)$  such that  $Y_t$  converges weakly to  $Y^*$  in finite dimensional distributions. Furthermore, with*

$$\alpha^* = \frac{K}{2} \mathbb{E}\left[\mathbb{P}\left[\max_{0 \leq s \leq t} W_s - Y^*(s) \leq 0 \mid Y^*\right]\right],$$

*we have that  $\frac{1}{t}X_t$  converges in probability to  $\alpha^* > 0$ .*

**Proof.** Since  $Y_t$  is stochastically decreasing it must converge in distribution to some limit  $Y^*$ . By Claim 2.2

$$\mathbb{P} \left[ \frac{K}{2} \mathbb{P}[\max_{0 \leq s \leq t} W_s - Y^*(s) \leq 0 \mid Y^*] \geq \frac{K}{10} 2^{-i/2} \prod_{i=i^*}^{\infty} (1 - e^{1 - \max\{1, 10i/\sqrt{2}\}}) \right] \geq \frac{1}{2},$$

and so  $\alpha^* = \lim_t \mathbb{E}S(t) > 0$ . To show convergence in probability fix  $\epsilon > 0$ . For some large enough  $L$ ,

$$\mathbb{E}[\frac{1}{L} X_L] = \frac{1}{L} \int_0^L \mathbb{E}S(t) dt \leq \alpha^* + \epsilon/2.$$

Let  $N_k = \mathbb{E}[X_{kL} - X_{(k-1)L} \mid \mathcal{F}_{(k-1)L}]$  and  $R_k = X_{kL} - X_{(k-1)L} - N_k$ . By monotonicity

$$N_k \leq \mathbb{E}[\frac{1}{L} X_L] \leq \alpha^* + \epsilon/2.$$

The sequence  $R_k$  are martingale differences with uniformly bounded exponential moments (since it is bounded from below by  $-(\alpha^* + \epsilon/2)$  and stochastically dominated by a Poisson with mean  $LK$ ). Thus

$$\lim_n \frac{1}{n} \sum_{k=1}^n R_k = 0 \text{ a.s. .}$$

It follows that almost surely  $\limsup_t \frac{1}{t} X_t \leq \alpha^*$ . Since  $X_t$  is stochastically dominated by  $\text{Poisson}(Kt)$  we have that  $\mathbb{E}[(\frac{1}{t} X_t)^2] \leq K^2 + K/t$  and so is uniformly bounded. Hence since  $\lim \mathbb{E} \frac{1}{t} X_t \rightarrow \alpha^*$  it follows that we must have that  $\frac{1}{t} X_t$  converges in distribution to  $\alpha^*$ .  $\square$

## 4 Regeneration Times

In order to establish almost sure convergence to the limit we define a series of regeneration times. We select some small  $\alpha(K) > 0$ , and say an integer time  $t$  is a regeneration time if

1. The function  $Y_t$  satisfies  $\inf_s Y_t(s) - y_\alpha(s) = 0$ .
2. For  $J_t$  the set of particles to the right of the aggregate at time  $t$ , their trajectories  $\{\zeta_j(s)\}_{j \in J_t}$  on  $(-\infty, t]$  satisfy

$$\inf_s \zeta_j(t-s) - (X_t - y_\alpha(s)) > 0.$$

Let  $0 \leq T_1 < T_2 < \dots$  denote the regeneration times and let  $\mathfrak{R}$  denote the set of regeneration times.

**Lemma 4.1** *For all  $K > 1$ , there exists  $\delta(K) > 0$  such that,*

$$\inf_{t \in \mathbb{N}} \mathbb{P}[t \in \mathfrak{R}] \geq \delta.$$

**Proof.** Let  $\mathcal{D}_t$  be the event that  $\inf_s Y_t(s) - y_\alpha(s) = 0$ . Provided that  $\alpha(K)$  is small enough by Lemmas 3.2 and 3.8 we have that

$$\mathbb{P}[\mathcal{D}_t] \geq \frac{1}{3}.$$

As the density of particles to the right of  $X_t$  is increasing in  $Y_t$  it is, therefore greatest when  $t = 0$  and so  $\mathbb{P}[t \in \mathfrak{R} \mid \mathcal{D}_t]$  is minimized at  $t = 0$ . Let  $w_\ell$  be defined as the probability

$$w_\ell = \mathbb{P}[\max_{0 \leq s \leq t} W_s - y_\alpha(s) > \ell]$$

For  $0 \leq \ell < \alpha^{-4}$  we simply bound  $w_\ell \leq 1$  so let us consider  $\ell \geq \alpha^{-4}$ . Then

$$\begin{aligned} w_\ell &\leq 1 - \mathbb{P}[M_\ell \leq \ell, \forall i \geq \lfloor \log_2(\ell) \rfloor : M_{2^{i+1}} \leq \ell + i2^{i+1}] \\ &\leq 1 - \mathbb{P}[M_\ell \leq \ell] \prod_{i \geq \lfloor \log_2(\ell) \rfloor} \mathbb{P}[M_{2^{i+1}} \leq \ell + i2^{i+1}] \\ &\leq 1 - (1 - e^{1-\ell^{1/2}}) \prod_{i \geq \lfloor \log_2(\ell) \rfloor} (1 - e^{1-i/\sqrt{2}}) \\ &\leq e^{1-\ell^{1/2}} + \sum_{i \geq \lfloor \log_2(\ell) \rfloor} e^{1-i/\sqrt{2}} \end{aligned}$$

where the third inequality is by the FKG inequality and the final inequality is by equation (3). Then we have that

$$\begin{aligned} \sum_{\ell \geq \alpha^{-4}} w_\ell &\leq \sum_{\ell \geq \alpha^{-4}} e^{1-\ell^{1/2}} + \sum_{\ell \geq \alpha^{-4}} \sum_{i \geq \lfloor \log_2(\ell) \rfloor} e^{1-i/\sqrt{2}} \\ &\leq \sum_{\ell \geq \alpha^{-4}} e^{1-\ell^{1/2}} + \sum_i 2^{i+1} e^{1-i/\sqrt{2}} < \infty, \end{aligned}$$

since  $2e^{-1/\sqrt{2}} < 1$ . Hence  $\sum_{\ell=0}^{\infty} w_\ell < \infty$  and so

$$\mathbb{P}[0 \in \mathfrak{R} \mid \mathcal{D}_0] = \mathbb{P}[\text{Poisson}(K \sum_{\ell=0}^{\infty} w_\ell) = 0] > 0.$$

Thus there exists  $\delta > 0$  such that  $\inf_{t \in \mathbb{N}} \mathbb{P}[t \in \mathfrak{R}] \geq \delta$ . □

We can now establish our main result.

**Proof.** [Theorem 1] By Lemma 4.1 there is a constant density of regeneration times so the expected inter-arrival time is finite. By Theorem 3.9 the process  $X_t$  travels at speed  $\alpha^*$ , at least in probability. By the Strong Law of Large Numbers for renewal-reward processes this convergence must also be almost sure. □

## 5 Higher dimensions

Our approach gives a simple way of proving positive speed in higher dimensions as well although not down to the critical threshold. Simulations for small  $K$  in two dimensions produce pictures which look very similar to the classical DLA model. Surprisingly, however, Eldan [1] conjectured that the critical value for  $d \geq 2$  is 0! That is to say that despite the simulations there is linear growth in of the aggregate for all densities of particles and that these simulations are just a transitory effect reflecting that we are not looking at large enough times. We are inclined to agree but our techniques will only apply for larger values of  $K$ . A better understanding of the notoriously difficult classical DLA model may be necessary, for instance that the aggregate has dimension smaller than 2.

Let us now assume that  $K > 1$ . In the setting of  $\mathbb{Z}^d$  it will be convenient for the sake of notation to assume that the particles perform simple random walks with rate  $d$  which simply speeds the process by a factor of  $d$ . The projection of the particles in each co-ordinate is then a rate 1 walk. We let  $U_t$  be the location of the rightmost particle in the aggregate (if there are multiple rightmost particles take the first one) at time  $t$  and let  $X_t$  denote its first coordinate. We then define  $Y_t(s)$  according to (1) as before. We call a particle with path  $(Z_1(t), \dots, Z_d(t))$  *conforming* at time  $t$  if  $Z_1(s) > X_s$  for all  $s \leq t$ . By construction conforming particles cannot be part of the aggregate and conditional on  $X_t$  form a Poisson process with intensity depending only on the first coordinate.

Let  $e_i$  denote the unit vector in coordinate  $i$ . The intensity of conforming particles at time  $t$  at  $U_t + e_1$  is then simply

$$K\mathbb{P}[\max_{0 \leq s \leq t} W_s - Y_t(s) \leq 0 \mid Y_t].$$

where  $W_s$  is an independent simple random walk. Similarly the rate at which conforming particles move from  $U_t + e_1$  to  $Y_t$  thus forming a new rightmost particle is

$$S(t) = \frac{1}{2}K\mathbb{P}[\max_{0 \leq s \leq t} W_s - Y_t(s) \leq 0 \mid Y_t],$$

the same as the formula we found in the one dimensional case. Of course by restricting to conforming particles we are restricting ourselves and so the rate at which  $X_t$  increments is strictly larger than  $S(t)$ . Since  $S(t)$  is increasing as a function of  $X_t$  (through  $Y_t$ ) we can stochastically dominate the one dimensional case by the higher dimensional process which establishes Corollary 2.

Let us now briefly describe how to improve upon  $K = 1$ . In the argument above we are being wasteful in two regards, first by only considering conforming particles and secondly by considering only a single rightmost particle. If there are two rightmost particles then the rate at which  $X_t$  increases doubles. The simplest way to get such a new particle is for a conforming particle at  $U_t + e_1 \pm e_i$  to jump first to  $U_t \pm e_i$  and then to  $U_t$ . There are  $(2d - 2)$  such location and the first move occurs at rate  $S(t)$  and the second at has probability  $1/(2d)$  to move in the correct direction and takes time exponential with rate  $d$ . After this sequence of events the rate at which  $X_t$  increments becomes  $2S(t)$ .

In Lemma 2.3, on which the whole proof effectively rests, we show that for  $\rho < 1$  if  $S(s) \geq \gamma$  for  $s \in [t, t + \Delta]$  then with exponentially high probability  $X_{t+\Delta} - X_t \geq \rho\Delta\gamma$  for any  $\rho < 1$  which is intuitively obvious since  $X_t$  grows at rate  $S(s) \geq \gamma$ . We can improve our lower bound on  $K$  by increasing the range of  $\rho$  for which this holds for small values of  $\gamma$ .

Define the following independent random variables

$$V_1 \sim \text{Exp}(\gamma), V_2 \sim \text{Exp}(\frac{2d-2}{2d}\gamma), V_3 \sim \text{Exp}(d), V_4 \sim \text{Exp}(\gamma)$$

where we interpret  $V_1$  as the time until the first conforming particle hits  $U_t$ . We will view  $V_2$  as the waiting time for a conforming particle to move from  $U_t + \pm e_i + e_1$  to  $U_t \pm e_i$  for some  $2 \leq i \leq d$  and we further specify that their next step will move directly to  $U_t$  which thins the process by a factor  $\frac{1}{2d}$ . Let  $V_3$  be the time until its next move. On the event  $V_2 + V_3 < V_1$  there is an additional rightmost particle before one has been added to the right of  $U_t$ . Now let  $V_4$  be the first time a conforming particle reaches this new rightmost site. So the time for  $X_t$  to increase is stochastically dominated by

$$T = \min\{V_1 + V_2 + V_3 + V_4\}.$$

Now using the memoryless property of exponential random variables,

$$\mathbb{E}T = \mathbb{E}V_1 - \mathbb{E}[(V_1 - (V_2 + V_3 + V_4))I(V_1 \geq V_2 + V_3 + V_4)] = \frac{1}{\gamma}(1 - \mathbb{P}[V_1 \geq V_2 + V_3 + V_4])$$

and

$$\begin{aligned} \mathbb{P}[V_1 \geq V_2 + V_3 + V_4] &= \mathbb{P}[V_1 \geq V_2]\mathbb{P}[V_1 \geq V_2 + V_3 \mid V_1 \geq V_2]\mathbb{P}[V_1 \geq V_2 + V_3 + V_4 \mid V_1 \geq V_2 + V_3] \\ &= \frac{\frac{2d-2}{2d}\gamma}{\gamma + \frac{2d-2}{2d}\gamma} \frac{d}{\gamma + d} \frac{\gamma}{2\gamma} \\ &= \frac{d-1}{2(2d-1)} \frac{d}{\gamma + d} \end{aligned}$$

In the proof we need only to consider the case where  $\gamma$  is close to 0 and

$$\lim_{\gamma \rightarrow 0} \gamma \mathbb{E}T = \frac{3d-1}{4d-2}.$$

Having  $X_t$  growing at rate  $\gamma \frac{4d-2}{3d-1}$  corresponds in the proof to linear growth provided that  $K > \frac{3d-1}{4d-2}$ . In the case for  $d = 2$  this means  $K > \frac{5}{6}$ . We are still being wasteful in several ways and expect that a more careful analysis would yield better bounds that tend to 0 as  $d \rightarrow \infty$ . However, we don't believe that this approach alone is sufficient to show that the critical value of  $K$  is 0 when  $d \geq 2$ . For that more insight into the local structure is likely needed along with connections to standard DLA.

## 6 Open Problems

In the one dimensional case the most natural open questions concern the behaviour of  $X_t$  for densities close to 1. Approaching  $K = 1$  from above one can ask what exponent does the speed of the process satisfy. Perhaps of most interest is what is the exponent of growth for  $X_t$  when  $K = 1$ . Heuristics suggest that it may grow as  $t^{2/3}$ .

In higher dimensions the main open problem is to establish Eldan's conjecture of linear growth for all  $K$ . Another natural question is to prove a shape theorem for the aggregate.

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